Area of Study 2 - Algebra

Linear Algebra

Linear algebra was heavily taught in years 8 to 10 in mainstream maths and units 1&2 maths methods. Only key aspects will be summarised here.

When solving linear equations for an unknown, e.g. *x*, it is important to rearrange the equation in such a way as to have all the unknowns on one side and the digits on the other. When solving equations with fractions, eliminate the denominators. See example below.

Example Question

Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

Solution

$$\frac{x}{5} - 2 = \frac{x}{3}$$
$$15\left(\frac{x}{5} - 2\right) = 15\left(\frac{x}{3}\right)$$
$$3x - 30 = 5x$$
$$-30 = 2x$$
$$x = -15$$

Notes

Remove the denominators first by multiplying by the lowest common multiple. In this case, the LCM is 15.

Solving simultaneous equations

The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs. Therefore, the preference is to algebraically solve the pair of simultaneous equations.

Example Question

Solve the equations $5x + 2y = 4$ and $3x - y = 6$ for the unknowns.

Solution using elimination

$$5x + 2y = 4\ (A)$$
$$3x - y = 6\ (B)$$

Multiple (B) by 2 to get (C)

$$6x - 2y = 12\ (C)$$

Add (A) and (C) to eliminate the variable y.

$$5x + 2y + (6x - 2y) = 4 + 12$$
$$11x = 16$$
$$x = \frac{16}{11}$$

Substitute x to find y.

$$3\left(\frac{16}{11}\right) - y = 6$$
$$\frac{48}{11} - \frac{66}{11} = y$$
$$y = -\frac{18}{11}$$

Solution using substitution

$$5x + 2y = 4\ (A)$$
$$3x - y = 6\ (B)$$

Rearrange (B) to make y the subject.

$$y = 3x - 6\ (C)$$

Substitute (C) into (A).

$$5x + 2(3x - 6) = 4$$
$$5x + 6x - 12 = 4$$
$$11x = 16$$
$$x = \frac{16}{11}$$

Substitute x to find y.
See solution at left.

Linear equation worded problems

Example Question 1 (Cambridge Senior Maths AC/VCE Specialist Mathematics 1&2)

An athlete trains for an event by gradually increasing the distance she runs each week over a five-week period. If she runs an extra 5 km each successive week and over the five weeks runs a total of 175 km, how far did she run in the first week?

Solution

Let x km be the distance that she runs in the first week.
1^{st} week = x km
2^{nd} week = x + 5 km
3^{rd} week = x + 10 km and so forth

$$x + (x+5) + (x+10) + (x+15) + (x+20) = 175$$
$$5x + 50 = 175$$
$$5x = 125$$
$$x = 25$$

Example Question 2 (Cambridge Senior Maths AC/VCE Specialist Mathematics 1&2)

Two tanks are being emptied. Tank A contains 100 litres of water and tank B contains 120 litres of water. Water runs from Tank A at 2 litres per minute, and water runs from tank B at 3 litres per minute. After how many minutes will the amount of water in the two tanks be the same?

Solution

Let t = number of minutes elapsed
Tank A: $100 - 2t = 0$
Tank B: $120 - 3t = 0$

Let $A = B$

$$100 - 2t = 120 - 3t$$
$$100 - 120 = -3t + 2t$$
$$-20 = -t$$
$$t = 20\text{min}$$

Example Question 3 (Cambridge Senior Maths AC/VCE Specialist Mathematics 1&2)

A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

Solution	Notes
Let x = amount of the 40% solution and y = amount of 15% solution. $700 = x + y$ (1) $0.24 \times 700 = 0.4x + 0.15y$ $168 = 0.4x + 0.15y$ (2)	Your two equation could be:
$168 = 0.4x + 0.15(700 - x)$ $168 = 0.4x + 105 - 0.15x$ $x = 252$	Substitute (1) into (2)
$700 = 252 + y$ $y = 448$	Substitute and solve for y.
Therefore 448 litres of the 15% solution and 252 litres of the 40% solution is required to make 700 litres of 24% solution.	

Literal Equations

Example Question

Transpose $\frac{ax+by}{c} = x - b$ to make x the subject.

Solution

$$\begin{aligned} \frac{ax+by}{c} &= x - b \\ ax + by &= xc - bc \\ by + bc &= xc - ax \\ b(y+c) &= x(c-a) \\ x &= \frac{b(y+c)}{c-a} \end{aligned}$$

Algebraic fractions

Example Question

Express $\frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x}$ as a single fraction.

Solution

$$\begin{aligned} \frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x} &= \frac{3x^3}{\sqrt{4-x}} + \frac{3x^2\sqrt{4-x}\sqrt{4-x}}{\sqrt{4-x}} \\ &= \frac{3x^3}{\sqrt{4-x}} + \frac{3x^2(4-x)}{\sqrt{4-x}} \\ &= \frac{3x^3 + 12x^2 - 3x^3}{\sqrt{4-x}} \\ &= \frac{12x^2}{\sqrt{4-x}} \end{aligned}$$

Notes

The square root multiplied by itself is equal to real number inside the surd. E.g.

$$\sqrt{a} \times \sqrt{a} = a$$

Example Question

Express $(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}}$ as a single fraction.

Solution

$$\begin{aligned} (x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}} &= (x-4)^{\frac{1}{5}} - \frac{1}{(x-4)^{\frac{4}{5}}} \\ &= \frac{(x-4)^{\frac{1}{5}}(x-4)^{\frac{4}{5}}}{(x-4)^{\frac{4}{5}}} - \frac{1}{(x-4)^{\frac{4}{5}}} \\ &= \frac{x-5}{(x-4)^{\frac{4}{5}}} \end{aligned}$$

The equation of the straight line

The general equation for a straight line is as follows:

$$y = mx + c$$

Where, $m =$ the gradient of the line and $c =$ the point where the line crosses the y-axis.

The gradient of the straight line

The gradient (m) represents the 'slope' of the line. A gradient can be positive or negative. It can also be undefined or equal to zero.

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\theta)$$

Midpoint of a line segment joining A to B

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Distance of a line segment between A to B.

$$\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example Question (VCAA, 2018 Exam 1 Question 7)

Let P be the point on the straight line $y = 2x - 4$ such that the length of OP, the line segment from the origin O to P, is a minimum.

a) Find the coordinates of P.

b) Find the distance OP. Express your answer in the form $\frac{a\sqrt{b}}{b}$, where a and b are positive integers.

Solution	Notes
Part a (method 1) The shortest distance from a fix point (in this case, from the origin) to any linear line is at right angles. In this case, we need to find the gradient of the tangent to $y = 2x - 4$. Therefore,	
$y_{tangent} = mx + c$ $= -\frac{1}{2}x + c$ $= -\frac{1}{2}x$	$c = 0$ because, the y-intercept is at (0,0)
Find the intersection of the tangent line from the origin to $y = 2x - 4$. $-\frac{1}{2}x = 2x - 4$ $x = \frac{8}{5}$	
Substitute x to find y. $y = 2x - 4$ $y = 2\left(\frac{8}{5}\right) - 4$ $y = -\frac{4}{5}$	
Therefore, the coordinate point $P = \left(\frac{8}{5}, -\frac{4}{5}\right)$	

Part a (method 2) The other method would be to use the distance formula and then find the derivative for the minimum value. Given $O = (0,0)$, $P = (x, 2x-4)$ and $\frac{d(OP)}{dx} = 0$ then $\overline{OP} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x)^2 + (2x-4)^2}$ $= \sqrt{5x^2 - 16x + 16}$	
$\frac{d(OP)}{dx} = \frac{5x-8}{\sqrt{5x^2-16x+16}}$ $0 = 5x - 8$ $x = \frac{8}{5}$	Differentiate either by rule or use the chain rule.
Substitute as above to find *y* and hence *P*. Part b	
$d = \sqrt{\left(0 - \frac{8}{5}\right)^2 + \left(0 + \frac{4}{5}\right)^2}$	Use the formula
$= \sqrt{\frac{80}{25}}$	
$= \frac{4\sqrt{6}}{5}$	Remember to give answer in required form

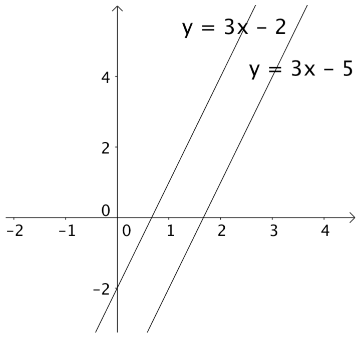
How many solutions questions

These questions always deal with linear equations and they usually involve you solving for m to make the pair of equations have no solution or one solution or infinite solutions.

No solution

Both linear equations have the same gradient and different y-intercepts

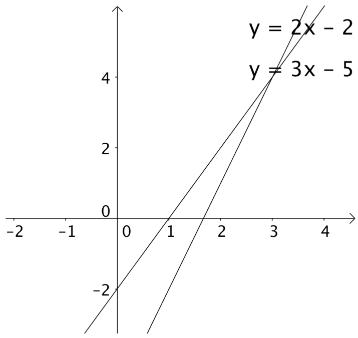
AKA: Parallel



One solution

Both linear equations have different gradients and different y-intercepts

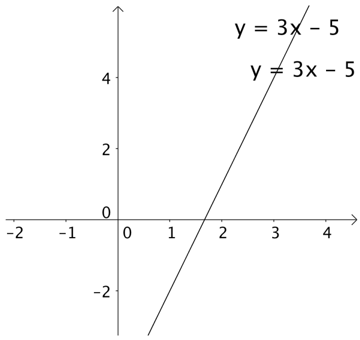
AKA: Intercept



Infinite solution

Both linear equations are exactly the same. They have the same gradient and y-intercept.

AKA: The same



Example Question (VCAA, 2014 Exam 2 Question 17)

Question 17

The simultaneous linear equations $ax - 3y = 5$ and $3x - ay = 8 - a$ have **no solution** for

A. $a = 3$

B. $a = -3$

C. both $a = 3$ and $a = -3$

D. $a \in R\backslash\{3\}$

E. $a \in R\backslash[-3,3]$

Step 1: Determine the conditions that make a pair of linear equations have no solution

Both linear equations have the same gradient and different y-intercepts.

Step 2: Rearrange the two equations to make y the subject

$$ax - 3y = 5$$
$$y_1 = \frac{1}{3}(ax - 5)$$
$$y_1 = \frac{ax}{3} + \frac{-5}{3}$$

$$3x - ay = 8 - a$$
$$y_2 = \frac{1}{a}(3x - 8 + a)$$
$$y_2 = \frac{3x}{a} + \frac{-8 + a}{a}$$

Step 3: Equate the gradients and intercepts as required

Gradients

$$\frac{a}{3} = \frac{3}{a}$$
$$a^2 = 9$$
$$a = \pm 3$$

y-intercepts

$$-\frac{5}{3} \neq \frac{-8 + a}{a}$$
$$-5a \neq -24 + 3a$$
$$-8a \neq -24$$
$$a \neq 3$$

Therefore, the answer is $a = -3$ (B).

Exam Tips: Many students confuse their negative and positive signs inside the equations. For example

$$y_2 = \frac{3x}{a} - \frac{8 + a}{a} \text{ (not correct!)}$$

Make sure that when you factorise the negative out to the front, that you factorise everything in the numerator as well. The correct equation is

$$y_2 = \frac{3x}{a} - \frac{8 - a}{a}$$

This is why, I prefer to put plus signs instead. Avoids this confusion and makes sure that the correct signs are always in front of a number.

Example Question (TSSM, 2010 Practice Guide Examination 2 Question 22)

Question 22

The value(s) of m such that the simultaneous equations $(m+1)x - 2y = 7$ and $3x + (2-m)y = 1$ have no solutions are

A. R^+

B. $m \in R^+$

C. $m = \frac{1+\sqrt{33}}{2}$ or $m = \frac{1-\sqrt{33}}{2}$

D. $m = 1 + \sqrt{33}$ or $m = 1 - \sqrt{33}$

E. $m = -1$

Step 1: Rearrange both equations to make y the subject

$$y = \frac{m+1}{2}x - \frac{7}{2} \qquad\qquad y = -\frac{3}{2-m}x + \frac{1}{2-m}$$

Step 2: For a set of equations to have no solution, they must have the same gradient and different y-intercepts.

Equating the two gradients

$$\frac{m+1}{2} = -\frac{3}{2-m}$$
$$(m+1)(2-m) = -6$$
$$2m - m^2 + 2 - m = -6$$
$$-m^2 + m + 2 = -6$$
$$m^2 - m - 2 = 6$$
$$m^2 - m - 8 = 0$$

Quadratic Formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$m = \frac{-(-1) \pm \sqrt{1 - 4(1)(-8)}}{2(1)}$$
$$m = \frac{1 \pm \sqrt{33}}{2}$$

Step 3: Check both values to ensure that you meet all the requirements (the y-intercepts are different)

Using your CAS, you will see that both m values will yield different y-intercepts.

Answer C.

The Quadratic Formula

Quadratic functions, when plotted, form a curve known as a parabola.
It is assumed in this course that the student has a firm grasp of quadratic **expansion** and **factorisation**; knows how to solve quadratic functions by inspection, completing the square and, if needed, the quadratic equation (below).

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The Quadratic Discriminant $\Delta = b^2 - 4ac$ (# of real solutions)

$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
There are no real solutions (except: there are two complex solutions)	There is one real solution	There are two real solutions

Example Question (VCAA, 2017 Exam 2 MC Question 7)

Question 7
The equation $(p-1)x^2 + 4x = 5 - p$ has no real roots when
A. $p^2 - 6p + 6 < 0$
B. $p^2 - 6p + 1 > 0$
C. $p^2 - 6p - 6 < 0$
D. $p^2 - 6p + 1 < 0$
E. $p^2 - 6p + 6 > 0$

Step 1: Recognise that this question relates to the discriminant. The answers should have been a giveaway. Rearrange the equation into standard form.

$$(p-1)x^2 + 4x + p - 5 = 0$$

Step 2: Determine when the quadratic would have no real roots.

$$\begin{aligned} b^2 - 4ac &< 0 \\ 4^2 - 4(p-1)(p-5) &< 0 \\ 16 - 4(p^2 - 6p + 5) &< 0 \\ -4p^2 + 24p - 4 &< 0 \\ -4(p^2 - 6p + 1) &< 0 \\ p^2 - 6p + 1 &> 0 \end{aligned}$$

Exam Tips: When dividing or multiplying an inequality by a negative number (i.e. -4) you must flip your inequality.

Long division & expression in a specific form

In recent years VCAA have expected students to be able to perform long division by hand and with technology. In most exams, there will be one or two questions which ask the student to express an expression into a specific form.

Example Question (VCAA, 2016 Exam 1 Question 4a)

Express $\frac{2x+1}{x+2}$ in the form $a + \frac{b}{x+2}$, where a and b are non-zero integers. 2 marks

Step 1: Express as a long division problem

$$\begin{array}{r|l} & 2 \\ x+2 & 2x+1 \\ & -(2x+4) \\ \hline & 0x-3 \end{array}$$

Step 2: Express your answer in the required form

$$y = \frac{2x+1}{x+2} = 2 - \frac{3}{x+2}$$

Indices & logarithmic rules

Review of index laws

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$

Review of rational indices

1. $a^{\frac{1}{n}} = \sqrt[n]{a}$
2. $\left(a^{\frac{1}{n}}\right)^n = a$
3. $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Review of logarithmic laws

- $log(ab) = log(a) + log(b)$
- $log\left(\frac{a}{b}\right) = log(a) - log(b)$
- $log\left(a^b\right) = b\, log\, a$
- $log_x(1) = 0$
- $log_a b = \frac{log_x(b)}{log_x(a)}$
- $log_x\left(\frac{1}{x^a}\right) = -a$

Example Question

Simply $\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}}$ into simplest form.

Solution

$$\begin{aligned}\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}} &= (a^3b^2)^{\frac{1}{3}} \div (a^2b^{-1})^{\frac{1}{3}} \\ &= \left(\frac{a^3b^2}{a^2b^{-1}}\right)^{\frac{1}{3}} \\ &= (ab^3)^{\frac{1}{3}} \\ &= b\sqrt[3]{a}\end{aligned}$$

Circular Function

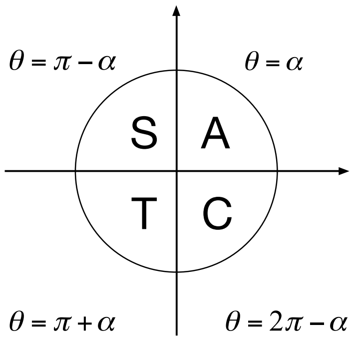
I have chosen to keep the graphing and algebraic components of circular functions together here for ease of reading.

Common exact value table

	0 degrees	30	45	60	90	180	270
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0	Undefined

Exam Tips: You must memorise your exact values – no ifs or buts. You can use the exact value triangles if it is easier for you.

Summary of signs (CAST)



Quadrant 1 (A): All positive

Quadrant 2 (S): Sine positive

Quadrant 3 (T): Tangent positive

Quadrant 4 (C): Cosine positive

Supplementary angles

You must memorise the exact values table. You will also notice that the CAST rule is implemented in this section too.

- In the second quadrant, sine is positive – therefore only the $\sin(\pi - \theta)$ conversion is positive
- In the third quadrant, tangent is positive – therefore only the $\tan(\pi + \theta)$ conversion is positive
- In the fourth quadrant, cosine is positive – therefore only the $\cos(2\pi + \theta)$ conversion is positive

Exam Tips: These questions usually appear as a give-away point in your non-calculator exam.

$$\sin(\pi - \theta) = \sin(\theta) \qquad \cos(\pi - \theta) = -\cos(\theta) \qquad \tan(\pi - \theta) = -\tan(\theta)$$

$$\sin(\pi + \theta) = -\sin(\theta) \qquad \cos(\pi + \theta) = -\cos(\theta) \qquad \tan(\pi + \theta) = \tan(\theta)$$

$$\sin(2\pi - \theta) = -\sin(\theta) \qquad \cos(2\pi - \theta) = \cos(\theta) \qquad \tan(2\pi - \theta) = -\tan(\theta)$$

Complementary angles

A lot of students have a love-hate relationship with complementary angles. The trick to mastering complementary angles is to understand your unit circle and the CAST technique. CAST tells you if a conversion should be positive or negative.

- In the first quadrant, all conversions are positive.
- In the second quadrant, sine is positive – therefore only the $\sin\left(\frac{\pi}{2}+\theta\right)$ conversion is positive
- In the third quadrant, tangent is positive – therefore only the $\cot\left(\frac{3\pi}{2}-\theta\right)$ conversion is positive
- In the fourth quadrant, cosine is positive – therefore only the $\cos\left(\frac{3\pi}{2}+\theta\right)$ conversion is positive

Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
$\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$	$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin(\theta)$	$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin(\theta)$	$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin(\theta)$
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$	$\sin\left(\frac{\pi}{2}+\theta\right)=\cos(\theta)$	$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos(\theta)$	$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos(\theta)$
$\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$	$\cot\left(\frac{\pi}{2}+\theta\right)=-\tan(\theta)$	$\cot\left(\frac{3\pi}{2}-\theta\right)=\tan(\theta)$	$\cot\left(\frac{3\pi}{2}+\theta\right)=-\tan(\theta)$

Summary of standard trigonometric functions

	$\text{a}\times\sin\,(nx+b)+c$	$\text{a}\times\cos\,(nx+b)+c$	$\text{a}\times\tan\,(nx+b)+c$
Amplitude	a	a	N/A
Period	$\frac{2\pi}{n}$	$\frac{2\pi}{n}$	$\frac{\pi}{n}$
Phase Shift	$-\frac{b}{n}$	$-\frac{b}{n}$	$-\frac{b}{n}$
Vertical Shift	c	c	c
Asymptotes	N/A	N/A	$x=\frac{\pi}{2n}-\frac{b}{n}$

Sine & cosine function graphs

The sine and cosine functions have the following general transformations.

$$y=a\sin\big(n(x+b)\big)+c \textbf{ or } y=a\cos\big(n(x+b)\big)+c$$

Where - a is the amplitude of the graph
- n is the dilation from the y-axis
- b is the horizontal translation (opposite to the sign)
- c is the vertical translation (the same as the sign)

The function has these properties.
- an amplitude from the centre line to the top of the graph.
- a period (or a cycle) before repetition. The general rule is $p=\frac{2\pi}{n}$

The cosine and sine graphs differ by a quarter of a period. Thus,

$$\sin\left(x-\frac{p}{4}\right)=\cos\,(x)$$

Where p is the period.

Finding the first tangent asymptote

Given $y = a\times\tan(nx+b)+c$, to find the first asymptote let $nx+b=\frac{\pi}{2}$ and solve for x. This works because the first asymptote of the function $y=\tan(x)$ is at $x=\frac{\pi}{2}$.

Example Question

Find all the asymptotes for the graph $y=-\tan 2\left(x-\frac{\pi}{4}\right)+1$ for $0\le x\le\pi$.

First asymptote:

$$2\left(x-\frac{\pi}{4}\right)=\frac{\pi}{2}$$
$$x-\frac{\pi}{4}=\frac{\pi}{4}$$
$$x=\frac{\pi}{2}$$

Period:

$$p=\frac{\pi}{n}=\frac{\pi}{2}$$

Find more asymptotes to fit inside your given domain

$$\text{asymptotes at } x=\frac{\pi}{2} \text{ and } x=\pi$$

Solving trigonometric questions with a specific domain

This section will have multiple worked examples on solving trigonometric questions.

Example Question (VCAA, 2017 Exam 2 MC Question 1)

Let $f: R\to R, f(x)=5\sin(2x)-1$.

The period and range of this function are respectively

A. π and $[-1,4]$
B. 2π and $[-1,5]$
C. π and $[-6,4]$
D. 2π and $[-6,4]$
E. 4π and $[-6,4]$

Solution

Period

$$p=\frac{2\pi}{n}=\frac{2\pi}{2}=\pi$$

Range

$$=[-1-5,-1+5]=[-6,4]$$

Example Question

Find all the solutions for the function $\sqrt{2}\cos(3x) + 2 = 3$ for $0 \le x \le 2\pi$.
Give your answer in exact values.

Solution

Step 1:

$$\sqrt{2}\cos(3x) + 2 = 3$$
$$\cos(3x) = \frac{1}{\sqrt{2}}$$

Notes: Rearrange your function

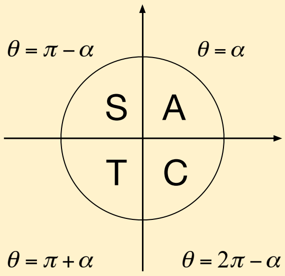
Step 2:

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(3x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

$$\alpha = \frac{\pi}{4}$$

Notes: Find the base angle (α) for that trig function

Step 3:



Therefore:

$$\theta = \alpha \qquad \theta = 2\pi - \alpha$$
$$3x = \frac{\pi}{4} \qquad 3x = 2\pi - \frac{\pi}{4}$$
$$x = \frac{\pi}{12} \qquad 3x = \frac{7\pi}{4}$$
$$x = \frac{7\pi}{12}$$

Notes: Determine which quadrant of the unit circle the solutions are in and solve for x.
Since $\cos(3x) = \frac{1}{\sqrt{2}}$ is positive, the solutions must be in the first and fourth quadrants (A and C).

Step 4:

$$p = \frac{2\pi}{n} = \frac{2\pi}{3}$$

Notes: Find the period

Step 5:

$$0 \le x \le \frac{24\pi}{12} \text{ and } p = \frac{8\pi}{12}$$

The answers are:

$$x = \frac{\pi}{12}, \frac{9\pi}{12}, \frac{17\pi}{12} \text{ or } x = \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$$

Finally, rearranged

$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Notes: Find all the solutions in the given domain by adding and subtracting the period
It's easier to convert the domain and period to have the same denominator as the solutions

Example Question (VCAA, 2017 Exam 12 MC Question 12)

The sum of the solutions of $\sin(2x) = \frac{\sqrt{3}}{2}$ over the interval $[-\pi, d]$ is $-\pi$.

The value of d could be

A. 0
B. $\frac{\pi}{6}$
C. $\frac{3\pi}{4}$
D. $\frac{7\pi}{6}$
E. $\frac{3\pi}{2}$

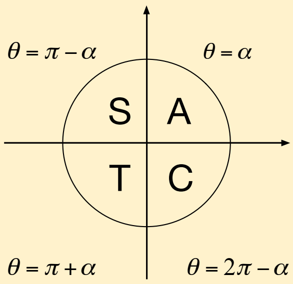
Solution

Step 1:

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(2x)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Notes: Find the base angle (α)

Step 2:



Therefore:

$$\theta = \alpha \qquad 2x = \frac{\pi}{3} \qquad x = \frac{\pi}{6}$$

$$\theta = \pi - \alpha \qquad 2x = \pi - \frac{\pi}{3} \qquad 2x = \frac{2\pi}{3} \qquad x = \frac{2\pi}{6}$$

Notes: Determine which quadrant of the unit circle the solutions are in and solve for x. Since $\sin(2x) = \frac{\sqrt{3}}{2}$ is positive, the solutions must be in the first and second quadrant (A and S).

Step 3:

$$p = \frac{2\pi}{n} = \pi$$

Notes: Calculate the period.

Step 4:

$$x = \frac{\pi}{6}, \frac{-5\pi}{6}, \frac{7\pi}{6} \text{ or } x = \frac{2\pi}{6}, -\frac{4\pi}{6}, \frac{8\pi}{6}$$

Therefore, rearranged

$$x = -\frac{5\pi}{6}, -\frac{4\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{7\pi}{6}, \frac{8\pi}{6}$$

Notes: By adding and subtracting the period from step 3 to the solutions in step 2, calculate the all solutions starting at $-\pi$ up until $\frac{3\pi}{2}$ (largest value in the answer).

Step 5:

$$-\frac{5\pi}{6} + -\frac{4\pi}{6} + \frac{\pi}{6} + \frac{2\pi}{6} = -\frac{6\pi}{6} = -\pi$$

Notes: By trial and error determine the answer

Step 6:

$$\therefore \frac{2\pi}{6} \leq d < \frac{7\pi}{6}$$

The only answer that satisfies this requirement is option C.

Notes: d can be equal to the upper limit $\left(\frac{2\pi}{6}\right)$ but needs to be smaller than the next solution $\left(\frac{7\pi}{6}\right)$

Example Question (VCAA, 2013 Exam 1 Question 4)

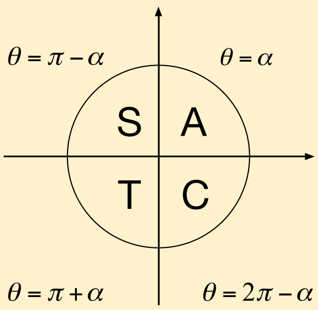
Question 4 (2 marks)

Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$

Step 1: Find the base angle (α)

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\left(\frac{x}{2}\right)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Step 2: Determine which quadrant of the unit circle the solutions are in and solve for x. Since $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$ is negative, the solutions must be in the third and fourth quadrant (T and C).



Therefore:

$$\theta = \pi + \alpha \qquad \frac{x}{2} = \frac{6\pi}{6} + \frac{\pi}{6} \qquad \frac{x}{2} = \frac{7\pi}{6} \qquad x = \frac{7\pi}{3}$$

$$\theta = 2\pi - \alpha \qquad \frac{x}{2} = \frac{12\pi}{6} - \frac{\pi}{6} \qquad \frac{x}{2} = \frac{11\pi}{6} \qquad x = \frac{11\pi}{3}$$

Step 3: Calculate the period

$$p = \frac{2\pi}{n} = 4\pi$$

Step 4: Convert your domain to have the same denominator as your solutions

$$\frac{6\pi}{3} \leq x \leq \frac{12\pi}{3}$$

Step 5: By adding and subtracting the period from the solutions in step 2, calculate all solutions within the given domain.

In this case, there are no more solutions within the given domain and your solutions in step 2 are the correct answers.

$$x = \frac{7\pi}{3}, \frac{11\pi}{3}$$

Example Question (TSSM, 2010 Exam Practice Guide Examination 2 Question 27)
Question 27

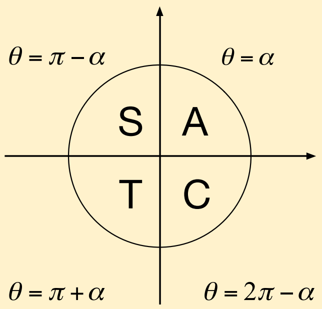
Consider the equation $2\sin\left(\frac{x}{2}\right) = -\sqrt{3}$. The 7th positive solution is

A. $\frac{56\pi}{3}$
B. $\frac{44\pi}{3}$
C. $\frac{80\pi}{3}$
D. $\frac{34\pi}{3}$
E. $\frac{82\pi}{3}$

Step 1: Find the base angle (α)

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\left(\frac{x}{2}\right)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Step 2: Determine which quadrant of the unit circle the solutions are in and solve for x. Since $\sin\left(\frac{x}{2}\right) = -\frac{\sqrt{3}}{2}$ is negative, the solutions must be in the third and fourth quadrant (T and C).



Therefore:

$$\theta = \pi + \alpha$$
$$\frac{x}{2} = \frac{3\pi}{3} + \frac{\pi}{3}$$
$$\frac{x}{2} = \frac{4\pi}{3}$$
$$x = \frac{8\pi}{3}$$

$$\theta = 2\pi - \alpha$$
$$\frac{x}{2} = \frac{6\pi}{3} - \frac{\pi}{3}$$
$$\frac{x}{2} = \frac{5\pi}{3}$$
$$x = \frac{10\pi}{3}$$

Step 3: Calculate the period

$$p = \frac{2\pi}{n} = 4\pi$$

Step 4: Convert your period to have the same denominator as your solutions

$$p = \frac{12\pi}{3}$$

Step 5: By adding the period from the solutions in step 2, calculate the first seven solutions

$$x = \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}, \frac{32\pi}{3}, \frac{34\pi}{3}, \frac{44\pi}{3}$$

Answer B.

General solutions to trigonometric equations

The syllabus stipulates that you need to know how to express trigonometric functions into its general form. However, in the 2016 and 2017 exams there were no questions that specifically asked you to express a trigonometric function into its general form. This section has no VCAA examination type questions.

- If $\sin(\theta) = a$, then $\theta = 2n\pi + \sin^{-1}(a)$ or $\theta = (2n+1)\pi - \sin^{-1}(a)$ where $a \in [-1,1]$ and $n \in Z$
- If $\cos(\theta) = a$, then $\theta = 2n\pi \pm \cos^{-1}(a)$ where $a \in [-1,1]$ and $n \in Z$
- If $\tan(\theta) = a$, then $\theta = n\pi \pm \tan^{-1}(a)$ where $a \in R$ and $n \in Z$