Area of Study 1 - Functions & Graphs

Basic definitions

Domain & Range

$$f: D \to R, f(x) = \text{[function]}$$

The domain (D) is the domain of the function is all the values of *x* for which the function exists.
The range is all possible values of *f(x)* over the domain.
The co-domain (R) is usually R and it contains the range of the function.

Exam Tips: remember your **implied domains**, e.g. $f(x) = \sqrt{x}$.
Since you cannot do a square root of a negative number in the real domain, it is implied that $x \geq 0$.

If you are asked to fully define a function, you must use the full notation above; it is not sufficient to just write the basic rule.

Odd & Even Functions

A function is odd when: $f(-x) = -f(x)$
E.g. consider $g(x) = \sin(x)$, sub in $-x$. $\therefore g(-x) = \sin(-x) = -\sin(x)$

A function is even when: $f(-x) = f(x)$
E.g. consider $h(x) = \cos(x)$, sub in $-x$. $\therefore g(-x) = \cos(-x) = \cos(x)$

Sums, difference and products

Sums and difference: $(f \pm g)x = f(x) \pm g(x)$
Product: $(f \times g)x = f(x) \times g(x)$

Both functions have domains at: $\text{Dom}(f) \cap \text{Dom}(g)$

Exam Tips: usually not important for exam 2 but might be a side question on exam 1 where you're asked to explicitly state the domain of a function.

Composite Functions

It is a function inside another function.
E.g. consider functions $f(x) = \cos(x)$ and $g(x) = \frac{1}{x}$, then
$f(g(x)) = f \circ g = cos\left(\frac{1}{x}\right)$ and $g(f(x)) = g \circ f = \frac{1}{\cos(x)}$

For $f(g(x))$ to be defined the $\text{Ran}(g) \subseteq \text{Dom}(f)$
If $f(g(x))$ exists then the $\text{Dom}(f(g(x))) = \text{Dom}(g)$

Example Question

Let $f: [0, \infty) \to R, f(x) = \sqrt{x}$ and $g: [0, \infty) \to R, g(x) = x^3 - 8$. Find $f(g(x))$ and state the implied domain.

	Domain	Range
$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$g(x) = x^3 - 8$	$[0, \infty)$	$[-8, \infty)$

In this configuration the composite function does not exist because
Ran$g(x)$ is not a $\subseteq$ Dom$f(x)$

We will need to restrict the domain of $g(x)$. The easiest way to do that is to pretend the composite function exists and work backwards, i.e.

$$f(g(x)) = \sqrt{x^3 - 8}$$

The composite function does not exist when:

$$x^3 - 8 < 0$$
$$x^3 < 8$$
$$x < 2$$

	Domain	Range
$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$g(x) = x^3 - 8$	$[2, \infty)$	$[0, \infty)$

Therefore, the composite function exists and the implied domain is $[2, \infty)$.

Example Question (VCAA, 2014 Exam 2 MC Question 13)

The domain of the function h, where $h(x) = \cos(\log_a(x))$ and a is a real number greater than 1, is chosen so that h is a one-to-one function.
Which one of the following could be the domain?

A. $\left(a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$
B. $(0, \pi)$
C. $\left[1, a^{\frac{\pi}{2}}\right]$
D. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right)$
E. $\left[a^{-\frac{\pi}{2}}, a^{\frac{\pi}{2}}\right]$

Step 1: This is a composite function, the domain of $h(x)$ is dictated by $\log_a(x)$. Determine a realistic domain where $\log_a(x)$ is a one-to-one function. E.g. $0 \le \log_a(x) \le \pi$
I'm using the answers as a guide, all of them have a π in the answers.

Step 2: Rearrange and solve for x.

$$a^0 \le x \le a^\pi$$
$$1 \le x \le a^\pi$$

Therefore, the most appropriate answer is C.

Inverse Functions

An inverse function will only exist for one-to-one graphs. If the graph is not one-to-one, it is possible to restrict the domain to allow an inverse to exist.

Algebraically: we swap the x and y values
Graphically: we perform a reflection along the line $y = x$

Dom $f^{-1}(x)$ = Ran $f(x)$ and Ran $f^{-1}(x)$ = Dom $f(x)$

Exam Tips: to obtain full marks you need to show a systematic process. These questions usually appear in graphing questions and exam 2 multiple choice section.

Example Question

Find the inverse of the function $f: [4, \infty) \rightarrow R, f(x) = \sqrt{x-4} + 2$.

Solution	Notes
Step 1: $x = \sqrt{y-4} + 2$	Let $y = f(x)$ and swap x and y
Step 2: $y = (x-2)^2 + 4$ $y = x^2 - 4x + 8$	Solve for y
Step 3: $\frac{dy}{dx} = 2x - 4$ $0 = 2x - 4$ $x = 2$	If the function is not a one-to-one function, restrict the domain. In this case, since the inverse is a quadratic function we can restrict the function to the turning point to make it one-to-one. Also remember the notes about the Dom $f^{-1}(x)$ = Ran $f(x)$ above. I have derived the function here to find the stationary point for the inverse.
Step 4: $f^{-1}: [2, \infty) \rightarrow R, f^{-1}(x) = x^2 - 4x + 8$	Write your answer in the appropriate form.

Example Question (VCAA, 2016 Exam 2 Question 4bi)

Let $f: R\backslash\{-2\} \rightarrow R, f(x) = \frac{2x+1}{x+2}$. Find the rule and domain of f^{-1}, the inverse function of f.

Solution	Notes
Step 1: $x = \frac{2y+1}{y+2}$ $xy + 2x = 2y + 1$ $y = \frac{1-2x}{x-2}$	Let $f(x) = y$ and inverse swap x and y. Solve for y.
Step 2: $y = -2 - \frac{3}{x-2}$	Place the answer in the form $y = A + \frac{B}{x-2}$ using long division
Step 3: Dom $f^{-1}(x)$ = Ran $f(x)$ Dom $f^{-1}(x) = R\backslash\{2\}$ $f^{-1}: R\backslash\{2\} \rightarrow R, f^{-1}(x) = -2 - \frac{3}{x-2}$	Determine your domain and give your answer in the required form. **Exam Tips:** It is always better to put your function into a form that you're familiar with – it makes it easier for you to deduce transformation.

Example Question (VCAA, 2016 Exam 2 Question 5)

Which one of the following is the inverse function of $g: [3, \infty) \to R, g(x) = \sqrt{2x - 6}$?

A. $g^{-1}: [3, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

B. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x - 6)^2$

C. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\frac{x}{2} + 6}$

D. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \frac{x^2+6}{2}$

Solution

Step 1:

$$x = \sqrt{2y - 6}$$
$$x^2 + 6 = 2y$$
$$y = \frac{x^2 + 6}{2}$$
$$g^{-1}(x) = \frac{x^2 + 6}{2}$$

Step 2:

	Domain	Range
$g(x)$	$[3, \infty)$	$[0, \infty)$
$g^{-1}(x)$	$[0, \infty)$	$[3, \infty)$

Dom $g^{-1}(x) = [0, \infty) \to R$

$g^{-1}: [0, \infty) \to R, g^{-1}(x) = \frac{x^2+6}{2}$ (D).

Notes

Let $g(x) = y$ and swap y and x

Determine your domain and give your answer in the required form.

Trifles

One-to-one graph test

Draw a horizontal line through the function, if it crosses the graph once = one-to-one. If a function is not one-to-one, we can restrict the domain to make it one-to-one.

Intersection of $g(x)$ and its inverse

Since $g^{-1}(x)$ is a reflection of $g(x)$ along the line $y = x$, to find the intersection of $g(x)$ and $g^{-1}(x)$ we **only need to find the intersection of $g(x)$ and the line $y = x$.**

Example Question

If $g(x) = (x - 2)^2 - 4$ where $x \geq 2$ and $g^{-1}(x)$ is the inverse function. Find the point(s) of intersection of $g(x)$ and $g^{-1}(x)$.

Step 1: Equate simultaneously and solve $g(x)$ and $y = x$

$$x = (x - 2)^2 - 4$$
$$x = x^2 - 4x + 4 - 4$$
$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$

Therefore, $g(x)$ intersect with $g^{-1}(x)$ at $x = 0$ and $x = 4$. Since $x \geq 2$, the answer is only $x = 4$.

Summary of basic positive functions in Maths Methods 3&4

Linear (x^1)	Dom: R Ran: R	2 intercepts
Quadratic (x^2)	Dom: R Ran: $[TP: y\text{-value}, \infty^+)$	
Cubic (x^3)	Dom: R Ran: R	Can have 2 TP or 1 point of inflection
Quartic (x^4)	Dom: R Ran: variable	Can look like a quadratic with a flattened base or have up to three turning points. May have a point of inflection if a perfect cubic is used.
Square Root $\left(x^{\frac{1}{2}}\right)$	Dom: R^+ Ran: R^+	Inverse of the quadratic function (with domain restriction).
Cube Root $\left(x^{\frac{1}{3}}\right)$	Dom: R Ran: R	S-shape in nature. Inverse of the cubic function.
Hyperbola $\left(x^{-1} = \frac{1}{x}\right)$	Dom: $R\backslash\{0\}$ Ran: $R\backslash\{0\}$ Asymptotes: $x, y = 0$	Two asymptotes.
Truncus $\left(x^{-2} = \frac{1}{x^2}\right)$	Dom: $R\backslash\{0\}$ Ran: $(0, \infty^+)$ Asymptotes: $x, y = 0$	Two asymptotes. Looks like a tree trunk (truncus sounds like trunk).
Exponential (a^x)	Dom: R Ran: $(0, \infty^+)$ Asymptote: $y = 0$ y-int: (0,1)	In maths, Euler's number (e) is used a lot. You will see e^x a lot.
Logarithmic $(\log_a(x))$	Dom: $(0, \infty^+)$ Ran: R Asymptote: $x = 0$ x-int: (1,0)	Euler's number also appears in logs a lot and it's called the natural log – $\log_e(x) = \ln(x)$

Power Functions

Odd positive integer — E.g. $y = x^3, y = x^5$
- Has point of inflection (base graph)

Even positive integer — E.g. $y = x^2, y = x^4$
- Has local minimum

Odd negative integer — E.g. $y = x^{-1} = \frac{1}{x}$
- Asymptotes at y = 0 and x = 0
- Domain and range = R\{0}

Even negative integer — E.g. $y = x^{-2} = \frac{1}{x^2}$
- Asymptotes at y = 0 and x = 0
- Domain = R\{0} Range = R+

$f^{-1}(x) = x^{\frac{1}{n}}$ is the **inverse function** of $f(x) = x^n$. Domain restriction of *f(x)* is usually required.

Transformations

There are three types of questions

1. You have the original function and are asked to apply a set of transformations to find the new function.
2. You have the new function and told the set of transformation that were applied and are asked to find the original function.
3. You have the original and the new functions and are asked to find the set of transformations that were used to go from original to new.

It is **very** important that you know what each and every value does to a function.

$$f(x) = af\left(\frac{x-c}{b}\right) + d$$

$$T: R^2 \to R^2: T\left[\begin{pmatrix} x' \\ y' \end{pmatrix}\right] = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

- Dilation by a factor of **a** from the x-axis (in the y direction)
- Dilation by a factor of **b** from the y-axis (in the x direction)
- Translation of **c** units in the positive x direction
- Translation of **d** units in the positive y direction
- If **a** or **b** is negative then there is a reflection in the x or y-axis respectively

This table summarises all the transformation in VCE

Dilation of factor *a* from *x*-axis	$(x, y) \to (x, ay)$
Dilation of factor *a* in the (or parallel to the) y-direction	$y = f(x) \to \frac{y}{a} = f(x)$
Dilation of factor *b* from *y*-axis	$(x, y) \to (bx, y)$
Dilation in the *x*-direction	$y = f(x) \to y = f\left(\frac{x}{b}\right)$
Reflection in x-axis	$(x, y) \to (x, -y)$
Reflection in *y*-axis	$(x, y) \to (-x, y)$
Translation of *c* units in positive *x*-axis	$(x, y) \to (x + c, y)$
Translation of *d* in *y*-axis	$(x, y) \to (x, y + d)$

Example Question (VCAA, 2015 Exam 2 Question 11)

The transformation that maps the graph of $y = \sqrt{8x^3 + 1}$ onto the graph of $y = \sqrt{x^3 + 1}$ is a

A. dilation by a factor of 2 from the y-axis
B. dilation by a factor of 2 from the x-axis
C. dilation by a factor of ½ from the x-axis
D. dilation by a factor of 8 from the y-axis
E. dilation by a factor of ½ from the y-axis

Solution

Step 1: $y' = \sqrt{8(x')^3 + 1}$ and $y = \sqrt{x^3 + 1}$

Step 2:

$$\sqrt{8(x')^3 + 1} = \sqrt{x^3 + 1}$$
$$8(x')^3 + 1 = x^3 + 1$$
$$(x')^3 = \frac{x^3}{8}$$
$$x' = \frac{x}{2}$$

Notes

Relabel your equations with nomenclature you are used to.

The y component and its image are not affected by this transformation since there are no outside dilation factors.

Equating x and its image results in a dilation factor by a factor 2 from the y-axis.

Example Question (VCAA, 2014 Exam 2 Question 12)

The transformation $T: R^2 \to R^2$ with rule $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ maps the line with equation $x - 2y = 3$ onto the line with equation

A. $x + y = 0$
B. $x + 4y = 0$
C. $-x - y = 4$
D. $x + 4y = -6$
E. $x - 2y = 1$

Solution

Step 1: Expand the transformation matrix.

$$T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} -x + 1 \\ 2y - 2 \end{bmatrix}$$

Step 2: Rearrange the transformation matrix and solve for x and y.

$$x' = -x + 1 \qquad y' = 2y - 2$$
$$x = 1 - x' \qquad y = \frac{1}{2}(y' + 2)$$

Step 3: Substitute the equations from step 2 into $x - 2y = 3$.

$$(1 - x') - 2\left(\frac{1}{2}(y' + 2)\right) = 3$$
$$1 - x' - y' - 2 = 3$$
$$-x' - y' = 4$$

Notes

I like to use $T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right)$ to represent the image instead of the one in the question.

You could continue to solve for y' but this is not necessary.

Example Question (VCAA, 2017 Exam 2 MC Question 10)

A transformation $T: R^2 \rightarrow R^2$ with rule $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps the graph of $y = 3\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ onto the graph of

A. $y = \sin(x + \pi)$
B. $y = \sin\left(x - \frac{\pi}{2}\right)$
C. $y = \cos(x + \pi)$
D. $y = \cos(x)$
E. $y = \cos\left(x - \frac{\pi}{2}\right)$

Solution

Step 1:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2x \\ \frac{1}{3}y \end{bmatrix}$$

$$x = \frac{1}{2}x' \qquad y = 3y'$$

Notes: Expand the transformation matrix and solve for x and y

Step 2:

$$3y' = 3\sin\left(2\left(\frac{1}{2}x' + \frac{\pi}{4}\right)\right)$$

$$y' = \sin\left(x' + \frac{\pi}{2}\right)$$

Notes: Substitute x and y into the original equation and solve for y'.

You will notice that none of our answer are correct. This means that a different technique was used to 'alter' the appearance of the answer.

By intuition we can eliminate options B and E. This leaves options A, C and D. Options C and D have cosine in the answer – can we make sine into cosine? Yes! Remember that cosine only differs from sine by a horizontal translation of a quarter of the period. Generally speaking, $\sin\left(x + \frac{\pi}{4}\right) = \cos(x)$.

Therefore, using the original equation in the question: $3\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$ becomes $3\cos(2x)$ and our new function is:

$$y = 3\cos(2x)$$

Step 3:

$$3y' = 3\cos\left(2\times\frac{1}{2}x'\right)$$

$$y' = \cos(x')$$

Notes: Substitute x and y (from step 1) into the new equation $y = 3\cos(2x)$.

Example Question (VCAA, 2016 Exam 2 MC Question 12)

Question 12

The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis followed by a dilation from the y-axis by a factor of $\frac{1}{2}$.

Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5 - 4x}$
B. $f(x) = -\sqrt{x - 5}$
C. $f(x) = \sqrt{x + 5}$
D. $f(x) = -\sqrt{4x - 5}$
E. $f(x) = -\sqrt{4x - 10}$

Solution

Step 1: Apply the first transformation (reflection in the x-axis)

$$f(x) = -\sqrt{2x - 5}$$

Step 2: Apply the second transformation (dilation from the y-axis by a factor of ½)

$$f(x) = -\sqrt{\frac{2}{\frac{1}{2}}x - 5}$$

$$f(x) = -\sqrt{4x - 5}$$

Hence, answer is D.

Matrices

Often used in mathematics to simplify complicated mathematics.

Exam Tips: you often find matrices used in transformation questions or when asked to solve a system of equations. They are often in the multiple choice section of exam 2.

Addition and subtraction	$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix}$
Multiplication by scalar	$3\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}3 & 6\\9 & 12\end{bmatrix}$
Multiplication of matrices (Only possible if the no. of columns of the first matrix is the same as the no. of rows of the second)	$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \times \begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}1\times5 + 2\times6\\3\times5 + 4\times6\end{bmatrix} = \begin{bmatrix}17\\39\end{bmatrix}$
The determinant of 2x2 matrix	$Det\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = ad - bc$ If the determinant $= 0$ then the inverse does not exist.
Inverse of 2x2 matrix	$\begin{bmatrix}a & b\\c & d\end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$
The identity matrix (Must be a square matrix and has a leading diagonal of 1s and 0s in all other elements)	$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ or $I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$
Multiplying a matrix by its inverse	Multiplying a matrix by its inverse is equal to the identity matrix $A\times A^{-1} = I$